Darker-than-black solitons: Dark solitons with total phase shift greater than π

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We discuss the stationary- and collisional-phase properties of dark optical solitons in a saturable nonlinear medium. Both the special case of an analytically solvable model of a saturable nonlinearity and other similar nonlinearities are considered. It is shown analytically that for a saturable nonlinearity the total phase shift across a single soliton may be much larger than π which is the maximum value in a Kerr medium and corresponds to a black soliton. Furthermore, for a given phase shift larger than π we find analytically that there exists two soliton solutions with different contrast. It is demonstrated numerically that these dark solitons which have an extraordinarily large total phase shift, propagate stably, and survive after collisions, maintaining almost the same large value of the phase shift. Additionally, we show that the stability criterion introduced earlier for multiple dark solitons is not sufficiently general and is effective only for some types of nonlinearity.

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I. INTRODUCTION

Since the pioneering work by Zakharov and Shabat [1] and Hasegawa and Tappert [2], dark optical solitons have been an active topic of research in nonlinear and fiber optics. The generation of dark solitons in optical fibers was first demonstrated by Emplit *et al.* [3] and confirmed by Weiner *et al.* [4]. The propagation of dark spatial solitons in self-defocusing materials has been experimentally shown in [5,6]. Recently increased interest in dark spatial solitons has become connected with their possible application in optical logic devices [7] and waveguide optics as, for instance, dynamic switches and junctions [8,9]. These latter applications are based on a fact that dark spatial solitons actually create waveguides in a self-defocusing medium. Some X- and Y-waveguide junctions as well as adiabatic couplers have been discussed and experimentally demonstrated.

In contrast to a bright soliton, which is a pulse on a zero-intensity background, a single dark soliton appears as an intensity dip in an infinitely extended constant background (quasi-plane-wave). Apart from the inverse intensity profile (in comparison with a bright soliton), an additional unique feature of a dark soliton is its specific phase profile. The dark-soliton phase chirp is a monotonic and odd function of the transverse (spatial) coordinate.

The total phase shift across the soliton is directly related to other soliton parameters such as its intensity contrast and propagation angle (transverse velocity) as well as the type of nonlinearity. Usually it is believed that the total phase shift across a dark soliton cannot exceed the value of π. This is certainly true for dark solitons in a Kerr medium as well as in the case of more sophisticated types of nonlinear media [10]. We have analytically found, however, that for saturable nonlinear media this restriction is no longer valid and in some cases the total phase shift can exceed not only the value π but even 2π. In principle, there is no upper limit to the total phase shift for dark solitons in saturable media.

One of the important physical consequences of our work, which can potentially rapidly increase the application of dark solitons, is the fact that the total phase shift across the soliton can be exactly 2π. It means that the soliton can be excited as a perturbation of the plane wave with the same phase at both sides of the soliton. Stability of this type of formation is also important for applications.

Besides, most of the new dark soliton phenomena that can be practically utilized require strong nonlinearities. Recent studies show that materials exhibiting strong nonlinear effects do not usually fall into the Kerr-type category. They typically exhibit saturation of the refractive index change, such as, for instance, semiconductor doped glasses [11]. Actually, even standard Kerr materials depart from the Kerr model with increasing light intensity. Also, thermal nonlinearity, which has been used successfully to demonstrate some applications of dark solitons, mimics the Kerr model for low light intensity only. This motivates detailed studies of soliton properties in media with saturable nonlinearity.

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This paper is organized as follows. In the next section, we will discuss phase properties of the few analytically solvable non-Kerr models for nonlinearity. Next, we present an analysis of the solitons of saturable nonlinearity. The subsequent section will deal with the stability of these solitons. We will end the paper with the summary.

II. SOLITON PHASE IN SOME ANALYTICALLY SOLVABLE NONLINEAR MODELS

In this work we are concerned with the soliton solution to the (normalized) nonlinear Schrödinger equation (NLSE) describing spatial beam propagation in self-defocusing nonlinear media:

$$iq_z+\tfrac{1}{2}q_{xx}-qf(I)=0 , \tag{1}$$

with z and x being the longitudinal and transverse coordinates, respectively. The function $f(I=|q|^2)$ describes the nonlinear refractive-index change as a function of light intensity I. $f(I)=I=|q|^2$ in the case of a Kerr nonlinearity. In this case a dark soliton is the solution of Eq. (1) in the form

$$q=q_0\{\sqrt{1-A^2}+iA\tanh[q_0A(x-vz)]\}e^{iq_0z} , \tag{2}$$

where q_0 is the background amplitude, A is a soliton parameter, and $v=q_0\sqrt{1-A^2}$ is its transverse velocity. The squared value of the parameter A coincides with the general definition of the intensity contrast:

$$A^2=\frac{I_0-I_1}{I_0} , \tag{3}$$

where $I_0=|q_0|^2$ is the background intensity and I_1 is the minimum intensity. The total phase shift is given by

$$\phi=2\arcsin(A) \tag{4}$$

and the propagation angle θ or transverse velocity $v=\tan\theta$ can be written in terms of ϕ and the background amplitude:

$$\tan\theta=q_0\cos(\phi/2) \tag{5}$$

The total phase shift ϕ varies between 0 and π. The larger the total phase shift, the darker the soliton (higher contrast). For the phase equal to π the soliton is black ($A^2=1$) and propagates perpendicularly to the initial wave front ($\theta=0$). As will be shown later, the π limitation for the soliton phase shift is not just a peculiarity of the Kerr model. However, it will also be shown that there is no restriction on the total soliton phase shift in some models for the nonlinear refractive-index change, which is more general than the Kerr case (particularly for saturable nonlinearity).

To clarify the main ideas of our work, we consider first a couple of simple examples of a more general function $f(I)$ describing the medium. These examples permit exact solutions of Eq. (1) enabling a full analysis of the soliton properties. We will be particularly interested in the phase properties of the dark solitons. The first model is an extended Kerr model [10,12] in which the function $f(I)$ is of the form

$$f(I)=I+\eta I^2 , \tag{6}$$

where the coefficient η can have either a positive or negative sign. As has been pointed out in [12], such a relation may describe for instance a medium doped with various dopants having different saturation parameters. When η is positive the refractive-index changes are stronger than for a Kerr medium. We will call this the case of "superlinear intensity dependence." In the case of $\eta<0$, the function $f(I)$ exhibits sublinear intensity dependence. For $\eta=0$, relation (6) describes the Kerr nonlinearity. The total phase shift of the dark soliton in this case is given by

$$\phi=2\arcsin\left[\frac{A}{A^2+(1-A^2)c}\right] , \tag{7}$$

where coefficient c is

$$c=\frac{1+2\eta I_0(4-A^2)/3}{1+2\eta I_0(1-A^2)/3} .$$

The second example that we consider is a threshold nonlinearity [13]. In this case, the refractive index changes as a step function if the light intensity exceeds some threshold value I_{crt}:

$$f(I)=\begin{cases}\delta & \text{for } I\geq I_{\text{crt}}\\ 0 & \text{otherwise} .\end{cases} \tag{8}$$

It is not a very realistic model, but it turns out to be very useful in explaining some fundamental properties of dark solitons [13]. The total phase shift of the dark soliton in this model is given by

$$\phi=\pi(1-\sqrt{1-A^2}) . \tag{9}$$

In order to illustrate the soliton phase properties resulting from these two models, we use the complex plane describing the real and imaginary parts of the field amplitude $q(x,z)$. Any solution of Eq. (1) in the form of a soliton can be represented on this plane by some trajectory (a parametric curve at fixed z with x as a parameter). Some examples of these trajectories for Kerr, extended Kerr, and step index saturation models are shown in Figs. 1(a), 1(b), and 1(c), respectively.

The background of the dark soliton in each case is a solution of Eq. (1) in the form of a plane wave:

$$q=q_0\exp[-if(q_0^2)z+i\phi_0] , \tag{10}$$

where ϕ_0 is the background phase. The phase ϕ_0 is arbitrary, but phases on the left and on the right of the dark soliton are related. Hence, the soliton background on the complex plane is located anywhere on a circle with radius q_0. For simplicity, we put $q_0=1$ in all our figures. The variation of the soliton amplitude and phase is displayed on the complex plane by a curve which is located inside the circle and which starts and ends on the circle. These two points correspond to the background amplitude at $x\rightarrow\infty$ and $x\rightarrow-\infty$. The total phase shift across the soliton profile is given by the angle subtended by this curve from the origin. For the Kerr nonlinearity depict-

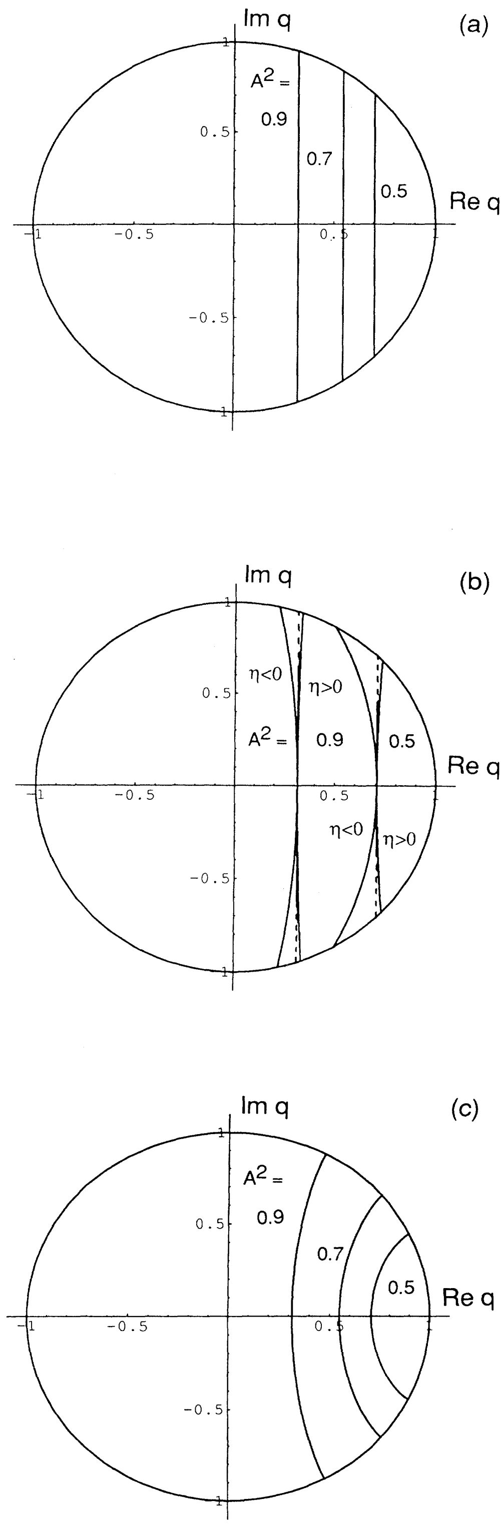

FIG. 1. Trajectories in the complex plane (Req, Imq) for dark solitons resulting from various nonlinear models: (a) Kerr-type nonlinearity, intensity contrast $A^2=0.5$, 0.7, and 0.9. (b) Extended Kerr model; various curves correspond to a different sign of the parameter η: $\eta<0$, $\eta>0$, and $\eta=0$ (dashed lines); intensity contrast $A^2=0.5$ and 0.9. (c) Threshold nonlinearity: intensity contrast $A^2=0.5$, 0.7, and 0.9.

ed in Fig. 1(a), the soliton is represented by a straight line [14]. The distance between the line and the origin corresponds to the minimum intensity. As the contrast A increases, the line moves towards the origin and the total phase shift of the soliton increases until it reaches π for a black soliton.

When the next-higher-order contribution is added to the Kerr nonlinearity [see Eq. (6)], the trajectory in the complex plane is no longer a straight line [Fig. 1(b)]. For the same contrast it has only one point in common with a straight line representing the Kerr case, which is a tangent to the curve. An interesting feature of this curve is that its curvature depends on the sign of the constant η. Let us define the sign of the curvature of the curve as positive if the curve bends inward (with respect to the origin) from the tangent line and as negative in the opposite case. Thus, the curve depicted in Fig. 1(b) has positive curvature when η is less than zero and negative curvature for positive η. Consequently, the total phase shift also depends on the sign of η. The position of the trajectory in general and its curvature depend also on the contrast A^2. As the contrast approaches unity (the case of a black soliton), the curvature decreases; the curve gradually becomes a straight line, and the total phase shift goes to π. Hence, π is the limiting case for the phase shift in the case of solitons with nonlinearity given by (6). This can also be seen from Eq. (7). The graph in Fig. 1(b) shows the relation between the curvature of the trajectory and the sign of η.

The case of a threshold nonlinearity [Eq. (8)] can be treated similarly. In this case, the trajectory in the complex plane always has negative curvature. An example of the trajectory in this case is shown in Fig. 1(c). Although the trajectory has negative curvature, as the contrast approaches unity the curvature decreases to zero so that the total phase shift again reaches π for the black soliton. The distinguishing feature of solitons in the models discussed above is that the total phase shift never exceeds π and this value is usually associated with the black soliton.

The question arises: Is it possible to have a total phase shift bigger than π? Obviously, the models of nonlinearity that lend negative curvature should always lead to a total phase shift lower than π. On the other hand, in cases with positive curvature there seems to be no apparent reason for a restriction on the total phase shift. In fact, as we show below, more general forms of the function $f(I)$ do lead to a total phase shift larger than π.

III. STATIONARY SOLITON SOLUTION OF THE SATURABLE NONLINEARITY

Let us consider, for example, the following model for a saturable nonlinearity [15]:

$$f(I)=\frac{I_s}{2}\left[1-\frac{1}{(1+I/I_s)^2}\right], \tag{11}$$

where I_s is a saturation parameter. As we mentioned earlier, it is of great practical importance to discuss saturable models for nonlinear refractive-index change, since real materials would much better fit under such models than for the Kerr type. We chose this particular form of the nonlinearity hoping that it would be a representative model of a "generic" saturable medium and because the corresponding nonlinear Schrödinger equation can be integrated analytically [16]. The latter feature would allow deeper and more exhaustive studies of the soliton properties than the numerical simulations alone.

The solution of Eq. (1) with nonlinear function (11) yields the following relation for the total soliton phase shift.

$$\phi=2\tan^{-1}(\beta/\mu)+2\mu\tanh^{-1}\beta \ , \tag{12}$$

where

$$\beta=\sqrt{I_0A^2/(I_0+I_s)}\ , \quad \mu=\sqrt{(1-A^2)I_0/I_s}\ ,$$

and I_0 is the background intensity. The total phase shift clearly depends on the saturation parameter I_s. Some examples of trajectories for dark solitons corresponding to the function (11) are shown in Fig. 2.

When the intensity is small compared with the saturation parameter, so that $I_0/I_s \ll 1$, the function $f(I)\approx I$ and the behavior of the trajectory in the complex plane approaches that of the Kerr model. For moderately weak saturation ($I_0/I_s<1$) the function (11) can be written as a series,

$$f(I)\approx I-\frac{3}{2I_s}I^2+\cdots \ , \tag{13}$$

and this is equivalent to the sublinear intensity dependence with negative η in Eq. (6). Hence, the trajectory in the complex plane is practically the same as that given by the extended Kerr model. It has a positive curvature, as we can see in Fig. 2(a). The larger the phase shift in this figure, the darker the soliton.

If the background intensity increases so that the function (11) enters the strong saturation region ($I_0 \gg I_s$), the total phase shift drastically increases and quite easily exceeds π. As shown in Figs. 2(b) and 2(c), solitons with a total phase shift larger than π or even 2π may exist. Note that when the total phase shift becomes larger than π, the trajectory bends around the origin rather than crossing it. This can be understood by noting that the refractive-index change close to the origin is larger than at finite $|q|$. Therefore we can expect that the energy required for the trajectory to cross the origin would be higher than that required to bend around it. Hence,

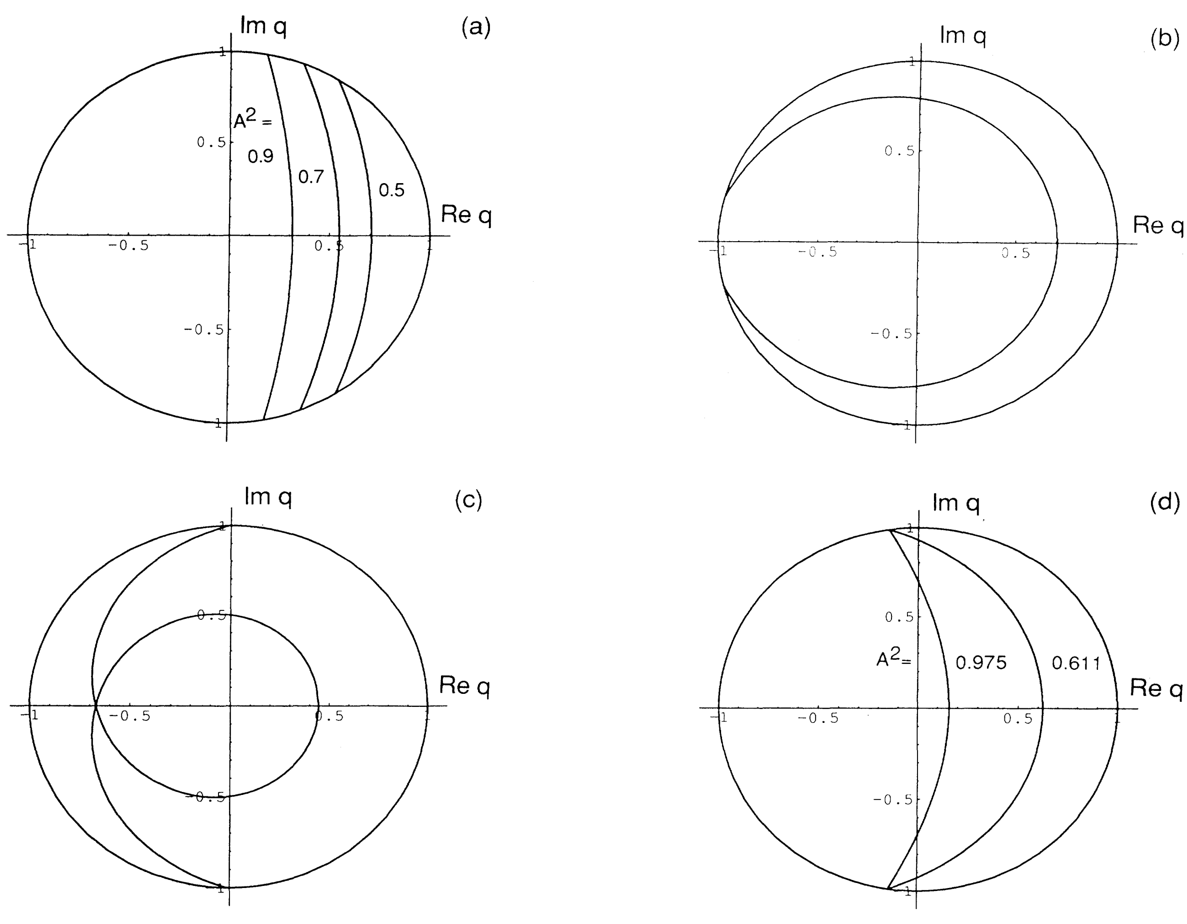

FIG. 2. Trajectories in the complex plane for dark solitons in a saturable nonlinear medium: (a) $I_0/I_s=1.0$ and the contrast $A^2=0.5$, 0.7, and 0.9; (b) $I_0/I_s=20$, $A^2=0.5$; (c) $I_0/I_s=50$, $A^2=0.8$; (d) $I_0/I_s=5$, $A^2=0.611$ and $A^2=0.975$.

these solitons do not reach the central zero and they cannot be black in this sense. On the other hand, if we stick with the standard terminology, which associates black solitons with the total phase equal to π, then the solitons with larger phase shifts could be considered "darker than black."

The π phase shift is, obviously, a special case. In a Kerr medium, π is simply the limiting value for the total phase shift. The variation of the shift with intensity contrast in a saturable medium for different values of the parameter I_0/I_s is shown in Fig. 3. The lowest curve in this figure corresponds to a Kerr nonlinearity as defined by Eq. (4). If the parameter I_0/I_s is less than 2.27, the curves are monotonic and the total phase shift is still less than π for any contrast. As soon as the parameter I_0/I_s exceeds the value 2.27, the total phase shift may become more than π in a certain range of contrasts to the left of $A=1$. Moreover, there are two values of the contrast corresponding to the same value of the total phase shift $\phi > \pi$. The maximum value of the total phase shift increases with increasing I_0/I_s and is, in principle, unlimited.

We can see now that two different dark solitons with different contrasts (and transverse velocities), but having exactly the same boundary conditions at infinity, exist. Trajectories in the complex plane corresponding to these two solitons are shown in Fig. 2(d). This type of soliton behavior could be called "soliton bistability" [10]. Both types of solitons are, indeed, stable, as we have found in our numerical simulations. Figure 4 shows the spatial intensity and phase profiles of these two solitons. It is worth noting that these solitons are rather wide. Due to saturation of the refractive-index change, their width increases significantly in order to weaken the effect of diffraction. For comparison, the intensity profile corresponding to a Kerr soliton with the same background intensity is also included. The contrast of the Kerr soliton is the same as for the soliton with the higher contrast in Fig. 2(d). We can see that the widths of the solitons in the saturable medium are much larger than in the Kerr medium.

Another special case (from the physical point of view) is $\phi=2\pi$. In this case, the background phases on both sides of the soliton can be considered equal. Hence, in this particular case, solitons can exist in the form of a perturbation of a plane wave. This means that using initial conditions with one intensity dip on the plane wave, it is possible to excite a single soliton rather than a pair of them, as in the case of the Kerr medium [14]. Special care must be taken to create the proper phase chirp to excite these solitons.

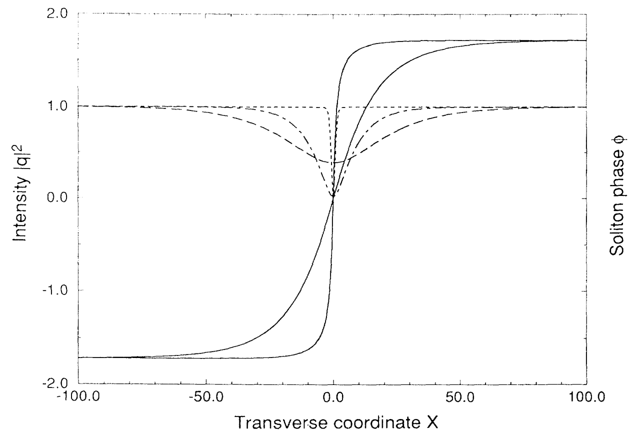

FIG. 4. Spatial phase and intensity profiles of dark solitons corresponding to two trajectories in Fig. 2(d). The long-dashed curve is the intensity profile of the soliton with lower contrast, while the dot-dashed curve is the intensity profile of the soliton with higher contrast. Two solid curves show the phase ϕ vs x for the same solitons. The dashed line is the intensity profile of a dark soliton in a Kerr medium. Its contrast $A^2=0.975$ is the same as for the soliton with the higher contrast in the saturable medium.

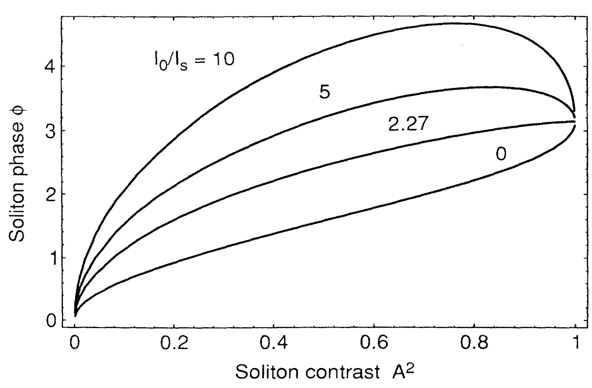

FIG. 3. Total phase shift ϕ vs intensity contrast A^2. The numbers above the curves quantify the parameter I_0/I_s.

IV. STABILITY OF THE LARGE-PHASE-SHIFT SOLITONS

We have studied the stability of dark solitons with large-phase shifts by numerically integrating the propagation Eq. (1). The split-step method combined with fast-Fourier transforms was used to simulate solutions of Eq. (1). The initial conditions corresponded to a pair of stationary dark solitons propagating towards each other. The stability problem can be subdivided into the problem of stability during free propagation (i.e. the stability of the stationary solutions) and the survival of solitons after collisions (robustness). In the most interesting high-saturation regime ($I_0/I_s > 2.27$), our simulations show that the solitons with large-phase shifts are indeed stable structures independent of the value of ϕ. During free propagation their profile is unchanged. This is the case even if the solitons have a small additional (not numerical) amplitude or phase perturbation. In Fig. 5 we present a few examples of propagation and collision of solitons with more than π-total phase shift, confirming these features.

The collision of solitons in a non-Kerr medium is usually inelastic, in the sense that it always leads to radiation during a collision. Radiation is also emitted from the impact area of the collision for large-phase-shift solitons.

As a result, the solitons change their parameters slightly after the collision. It is important to note that solitons after the collision maintain their total phase shift higher than π. This describes the general behavior, and the quantitative changes of the soliton parameters depend on their contrast. At fixed I_s the radiation (and hence the amount the parameters change) is practically negligible for small A^2, but increases for solitons with a contrast close to unity [compare Fig. 5(a) with Fig. 5(b) and Fig. 5(c) with Fig. 5(d)]. The amount of radiation increases with deeper saturation, i.e., with decreasing I_s [compare Fig. 5(d) with Fig. 5(b)]. The robustness of these solitons

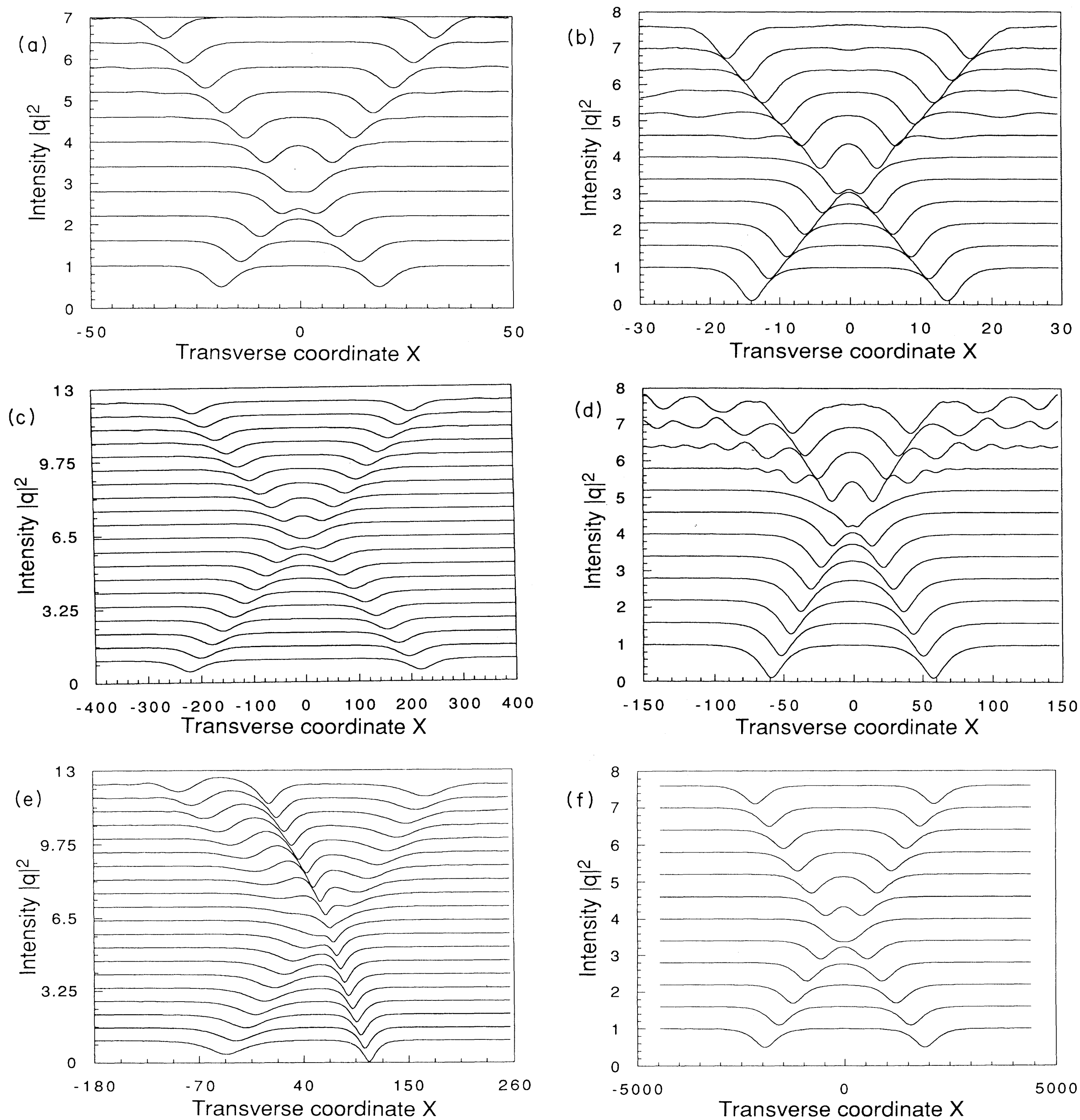

FIG. 5. Collision of dark solitons in a medium with saturable nonlinearity: (a) $A^2=0.5$, $I_s=1.0$; (b) $A^2=0.9$, $I_s=1.0$; (c) $A^2=0.5$, $I_s=0.2$; (d) $A^2=0.9$, $I_s=0.2$; (e) $A^2=0.611$ (for the soliton on the right) and 0.975 (for the soliton on the left), $I_s=0.2$; (f) $A^2=0.5$, $I_s=0.05$.

indicates that practically all relevant features of the soliton-based waveguide systems such as X and Y junctions found in Kerr material will be preserved, even if the medium exhibits strong saturation.

The special case of the collision of two solitons with the same total phase shift (but with different contrasts) is shown in Fig. 5(e). Even in this case, the solitons pass through each other, keeping almost the same large total phase shifts. The contrasts and angles of propagation are changed slightly after the collision, but the solitons themselves are stable and robust. The radiation is emitted asymmetrically mostly towards the side of the soliton with the greater contrast. The robustness of large-phase solitons is also evident from Fig. 5(f), which shows the collision of two moderate contrast solitons in the case of very strong saturation ($I_0/I_s=20$). Both solitons have the same phase shift (approximately 1.8π). Their propagation is stable and the collision produces virtually no radiation. An interesting feature of soliton collisions in the presence of saturation is the change of transverse velocities of the solitons, clearly evident in Figs. 5(b), 5(d), and 5(f). This is related to inelasticity of the collision process. A single soliton always has a well-defined transverse velocity which depends on the soliton parameters (background, contrast). Since these change during a collision, so too must the transverse velocity.

The numerical simulations demonstrate that these solitons are always stable and, hence, we cannot deduce any criterion that governs their stability. Mulder and Enns [17] suggested $\partial P/\partial A>0$ (where P is the complementary power) as a criterion for the stability of dark solitons in a medium with generalized nonlinearity. This criterion certainly does not work in our case. Figure 6 shows the complementary power

$$P=\int_{-\infty}^{\infty}(q_0^2-|q(x,t)|^2)dx \tag{14}$$

versus A^2 for our dark solitons with fixed $I_0=1$. This curve is monotonic for $I_s>1.8$ and has a single maximum at $I_s<1.8$. Hence, using Enns and Mulder's stability criterion $\partial P/\partial A>0$, the solitons with higher contrast to the right of the maximum of P should be unstable. This is not the case, as our numerical simulations show. Hence, we do not see any apparent connection between $\partial P/\partial A$ and soliton stability. Figure 5(e) shows the propagation and collision of two solitons, both having $\partial P/\partial A<0$ (and therefore potentially unstable according to the $\partial P/\partial A$ criterion). However, both solitons propagate before the collision without any distortions. The collision leads to some visible deformation of one of the solitons (with higher contrast), leaving the second one practically unchanged.

It is worth mentioning that the curve $P(A)$ becomes a function with a single maximum at a value I_0/I_s, which is different from 2.27 when the curve $\phi(A^2)$ also becomes a function with a single maximum. The values of A where these two curves have maxima are also different. Hence, no apparent connection exists between the stability of the solitons and the properties of any of the curves in Fig. 3 or in Fig. 6.

The large total phase shift solitons are stable not only relative to the perturbations of the solution but relative to a small perturbation of the equation as well. We performed numerical simulations of solutions of Eq. (1) with a small-loss term (to ensure adiabaticity of absorption process) added into the right-hand side of the equation. In these simulations, the solitons kept their shape and large total phase shift during propagation, provided the background intensity was well above the saturation level. We remind the reader that only in this case can the total phase shift exceed the value π. The propagation angle of the soliton and its other parameters changed smoothly during propagation in the presence of loss [18].

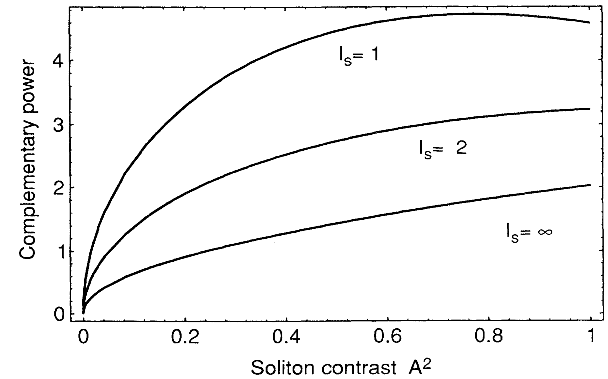

FIG. 6. Complementary power [Eq. (14)] vs contrast for $I_s\rightarrow\infty$, $I_s=2$, and $I_s=1.0$. Background intensity I_0 is equal to 1.

The solutions presented so far result from assuming a particular model of the nonlinearity [Eq. (11)]. It turns out, however, that they represent the general behavior of qualitatively similar models for the non-Kerr nonlinear medium. In our numerical simulations we also used a different saturation model (two-level atom model) and found phase and collisional properties similar to those for the model described by Eq. (11). This is important because it justifies our choice of nonlinearity as a model for the saturable medium.

V. SUMMARY

In summary, we have studied the phase properties of dark solitons using a model for a saturable nonlinearity which has an exact analytical solution. We showed that the total soliton phase shift may become bigger than π for strong saturation. For phase shifts higher than π, two solitons with different contrasts exist. We showed that, unlike the Kerr case, when a single dark soliton can be only created from odd initial conditions, a saturable medium actually permits excitation of one dark pulse as a perturbation to the plane wave. This effect has not been found previously. We also showed that solitons with large total phase shift are stable during propagation and survive after collisions, but with slightly changed parameters.

From a practical point of view, our results indicate that the nonlinear materials with strong nonlinearities, which often exhibit a saturation effect, are still good candidates for dark soliton applications.

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